# Math 142 Lecture 25 Notes

Daniel Raban

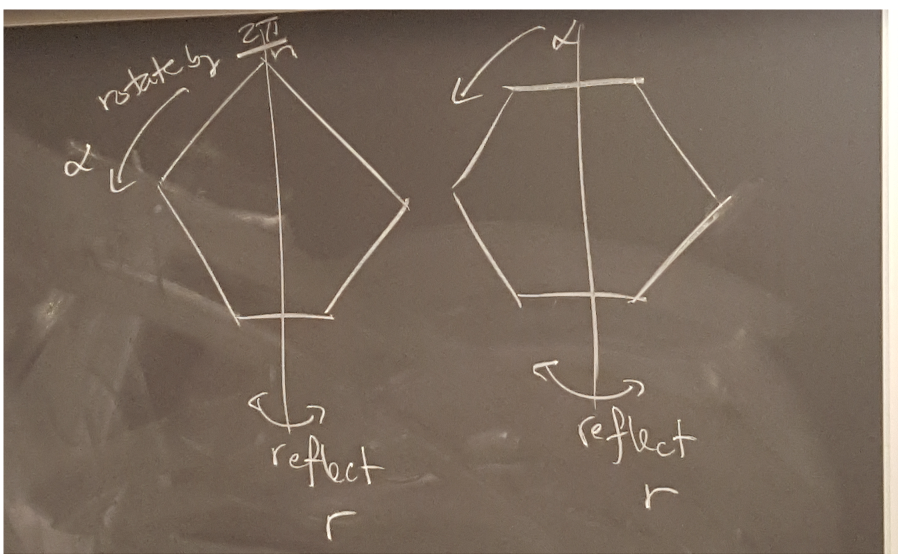
April 24, 2018

# 1 Knot Colorings in Algebraic Topology

## 1.1 Knot groups

Our goal is to relate $n$-colorings to algebraic topology. We will show that $n$-colorings of $K$ correspond (almost) to homomorphisms $\pi_1(\mathbb{R}^3 \backslash K) \to D_{2n}$; we will actually be overcounting by $n$. $D_{2n}$ is the group of symmetries of a regular $n$-gon,[1]

$$D_{2n} = \left\langle r, \alpha \mid \alpha r = r\alpha^{-1}, r^2 = 1, \alpha^n = 1 \right\rangle.$$

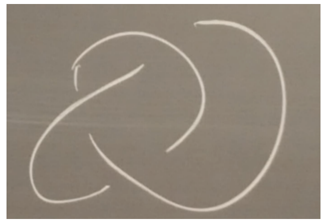



We will work out $\pi_1(\mathbb{R}^3 \setminus K)$ to be the following.

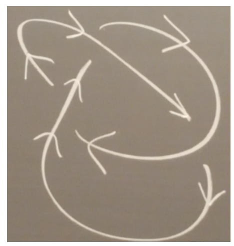
**Definition 1.1.** The *knot group* of $K$ is constructed by:

[1] Some people call this group $D_n$. You should always be clear with your notation when discussing this group.

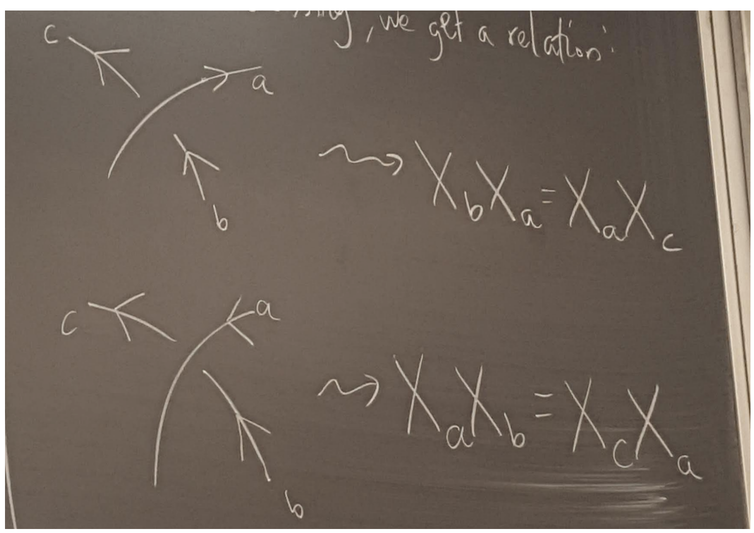
1. Take a nice projection of $K$.



2. Put a direction on the knot.



3. Number the arcs $1, \ldots, n$.
4. $\pi_1(\mathbb{R}^3 \setminus K)$ is generated by $x_1, \ldots, x_n$ (one for each arc).
5. For each crossing, we get a relation:



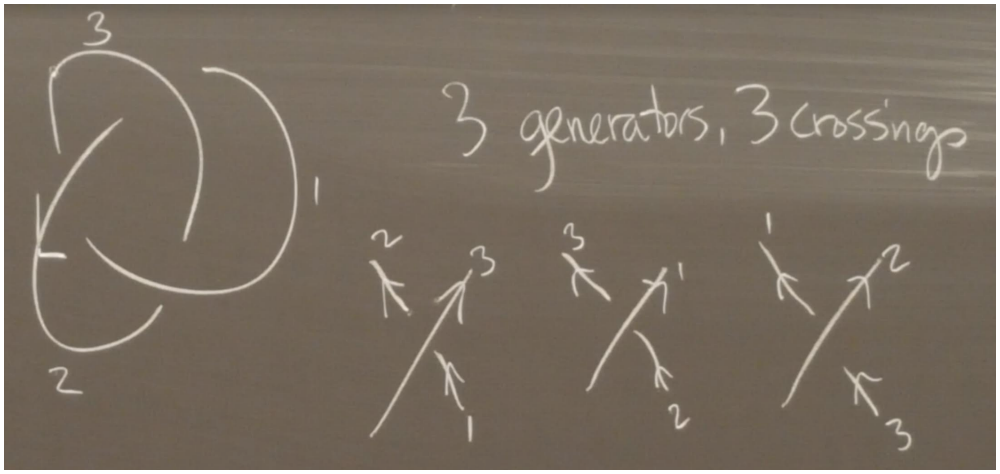

This says that $x_c$ is a conjugate of $x_b$.

We can also define this for oriented links, but we will not prove that here.

**Example 1.1.** Let $K$ be the unknot. $K$ has one arc and no crossings, so

$$\pi_1(\mathbb{R}^3 \setminus K) \cong \langle x_1 \rangle \cong \mathbb{Z}.$$

**Example 1.2.** Let $K$ be the trefoil knot. This has 3 arcs and 3 crossings.




$$\pi_1(\mathbb{R}^3 \setminus K) \cong \langle x_1, x_2, x_3 \mid x_1x_3 = x_3x_2, x_2x_1 = x_1x_3, x_3x_2 = x_2x_1 \rangle$$

Note that $x_3 = x_2x_1x_2^{-1}$, so $x_3$ is a redundant generator.

$$\cong \left\langle x_1, x_2 \mid x_1x_2x_1x_2^{-1} = x_2x_1x_2^{-1}x_2, x_2x_1 = x_1x_2x_1x_2^{-1} \right\rangle$$

Write $a = x_1$, $b = x_2$, and simplify.

$$\cong \langle a, b \mid aba = bab, bab = aba \rangle$$

These relations are redundant.

$$\cong \langle a, b \mid aba = bab \rangle .$$

In general, it is hard to tell apart groups like this by their generators. Before, we used Abelianization to tell apart fundamental groups. However, that approach doesn't work here.

**Proposition 1.1.** *Let $K$ be a knot. Then*

$$\mathrm{Ab}(\pi_1(\mathbb{R}^3 \setminus K)) \cong \mathbb{Z}.$$

*Proof.* Every relation is $x_a x_b = x_b x_c$. This reorders to $x_a x_b = x_c x_b$, which gives us that $x_a = x_c$. Check that all generators are identified this way. $\square$

Regardless, we are looking at the right object.

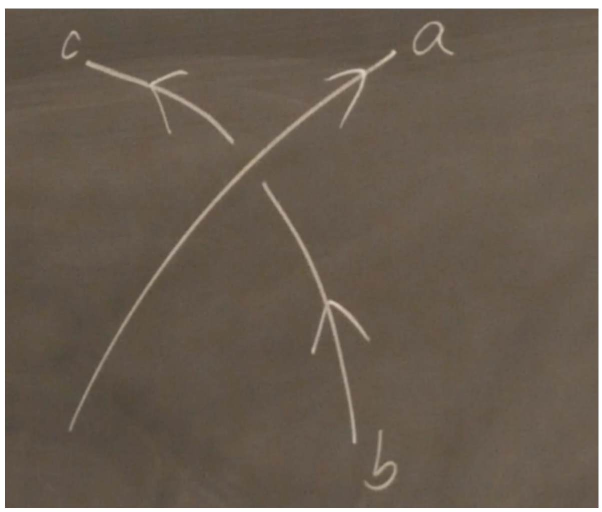
**Theorem 1.1** (Gordon-Luecke, 1989)**.** *$\pi_1(\mathbb{R}^3 \setminus K_1) \cong \pi_1(\mathbb{R}^3 \setminus K_2)$ iff $K_1$ and $K_2$ are equivalent.*

So this fundamental group determines the knot up to isotopy or mirroring.

## 1.2 Correspondence between knot colorings and fundamental group homomorphisms

**Theorem 1.2.** *Let $K$ be a knot. Then there is a correspondence between $n$-colorings of $K$ and homomorphisms $\pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$ (except for $n$ homomorphisms).*

*Proof.* Given an $n$-coloring that sends arc $i$ to color $\ell_i \in \{1, \dots, n\}$, construct the homomorphism $\pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$, via $x_i \mapsto r\alpha^{\ell_i}$. We need to check the relations. At a crossing



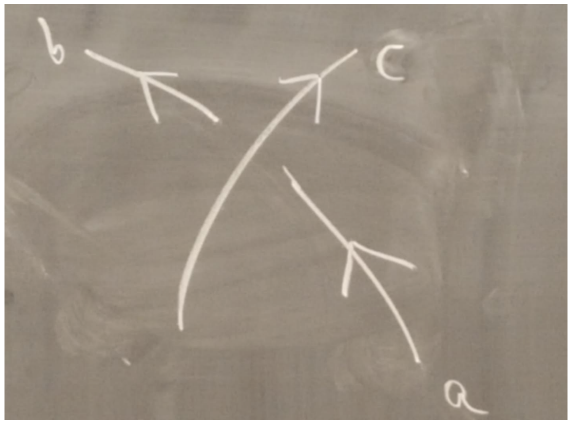

we know that $x_b x_a = x_a x_c$. We get

$$x_b x_a \mapsto r\alpha^{\ell_b} r\alpha^{\ell_a} = rr\alpha^{-\ell_b}\alpha^{\ell_a} = r^2\alpha^{\ell_a - \ell_b} = \alpha^{\ell_a - \ell_b},$$

$$x_a x_c \mapsto r\alpha^{\ell_a} r\alpha^{\ell_c} = rr\alpha^{-\ell_a}\alpha^{\ell_c} = r^2\alpha^{\ell_c - \ell_a} = \alpha^{\ell_c - \ell_a}.$$

We want $\alpha^{\ell_a - \ell_b} = \alpha^{\ell_c - \ell_a}$; i.e. we need $\ell_a - \ell_b \equiv \ell_c - \ell_a \pmod{n}$. This is equivalent to $2\ell_a \equiv \ell_b + \ell_c \pmod{n}$, which is the requirement for an $n$-coloring.

The argument is similar for the other crossing type. So we have a homomorphism. We can also go backward (homomorphism to coloring), but only if for all $i$, $\phi(x_i) = r\alpha^{\ell_i}$ for some $\ell_i$; $\ell_i$ will be the color of arc $i$.If $\phi(x_a) = \alpha^{\ell_i}$ for some $a$, we need $\phi(x_a x_c) = \phi(x_c x_b)$.




Count how many reflections we have on the left hand side and on the right.

| LHS # refls. | $\phi(x_c)$ |
|---|---|
| 1 | $r\alpha^{\ell_c}$ |
| 0 | $\alpha^{\ell_c}$ |

| RHS # refls. | $\phi(x_c)$ | $\phi(x_b)$ |
|---|---|---|
| 0 | $r\alpha^{\ell_c}$ | $r\alpha^{\ell_b}$ |
| 1 | $r\alpha^{\ell_c}$ | $\alpha^{\ell_b}$ |
| 1 | $\alpha^{\ell_c}$ | $r\alpha^{\ell_b}$ |
| 0 | $\alpha^{\ell_c}$ | $\alpha^{\ell_b}$ |

In either case, $\phi(x_b) = \alpha^{\ell_b}$ for some $\ell_b$. Follow our knot around, doing the same analysis at every crossing. Then $\phi(x_i) = \alpha^{\ell_i}$ for all $i$.

Now $\phi(x_a x_b) = \phi(x_c x_b)$ iff $\alpha^{\ell_a}\alpha^{\ell_c} = \alpha^{\ell_c}\alpha^{\ell_b}$. This is the condition that $\ell_a \equiv \ell_b \pmod{n}$. Check that this makes $\phi(x_i) = \phi(x_j)$ for al $i, j$. So ignore these homomorphisms (there are $n$ of them). Hence,

$$|\{n\text{-colorings of } K\}| = |\{\text{homomorphisms } \pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}\}| - n. \qquad \square$$

Here is an application of this result.

**Example 1.3.** Let $K$ be a knot siting on a torus that gives the element $3, 4 \in \pi_1(T^2)$; this goes 3 times around the torus and 4 times through the center hole. We can show that $\pi_1(\mathbb{R}^3 \setminus K) \cong \langle a, b \mid a^3 = b^4 \rangle$.

$K$ is not $n$ colorable for any $n$. If $\phi : \pi_1(\mathbb{R}^3 \setminus K) \to D_{2n}$ is a homomorphism such that $\phi(a) = r\alpha^i$ and $\phi(b) = r\alpha^j$, we need that $\phi(a)^3 = \phi(b)^4$. But $\phi(a)^3 = r\alpha^i r\alpha^i r\alpha^i = r\alpha^i$ and $r\alpha^j r\alpha^j r\alpha^j r\alpha^j = 1$. So we have a contradiction.